Teoremi dell'aritmetica di Robinson

e complemento dei capitoli 8 e 9

di Peter Smith

"An Introduction to Gödel's theorems"

Cambridge U.P 2007

$$\bar{m} + \bar{n} = \overline{m + n}$$

Ghilardi prop 31

Per induzione su n

$$n = 0 \Rightarrow \bar{m} + 0 = \bar{m} = \overline{m + 0}$$

IH $\quad \bar{m} + \bar{k} = \overline{m + k}$

$$\bar{m} + \overline{k+1} = S(\bar{m} + \bar{k}) = S(\overline{m+k}) = \overline{m+k+1}$$

Corollario $\quad 0 + \bar{n} = \overline{0 + n} = \bar{n}$

$$\bar{m} \times \bar{n} = \overline{m \times n}$$

$$\bar{m} \times 0 = \bar{0} = \overline{m \times 0}$$

IH $\quad \bar{m} \times \bar{k} = \overline{m \times k}$

$$\bar{m} \times \overline{k+1} = (\bar{m} \times \bar{k}) + \bar{m} = \overline{m \times k} + \bar{m} = \overline{(m \times k) + m} = \overline{m \times (k+1)}$$

$m \neq n \longrightarrow \overline{m} \neq \overline{n}$ Ghilardi prop 3.1

Dim Supponiamo che $m \neq n$ & $\overline{m} = \overline{n}$

Sia $m > n$ e $(m-n) = j+1$

$\overline{m} = \overline{n}$

$S^m 0 = S^n 0$

$S^{m-1} 0 = S^{n-1} 0$

$\vdots$

$S^{m-n} 0 = S^{n-n} 0$

$S^{j+1} 0 = S^0 0$

$S S^j 0 = 0$

↯ axiom 1

$\boxed{\text{Convenzione notazionale}}$

$\overline{m} = S^m 0$

$S^1 0 = S0$

$S^0 0 = 0$

$S^{m+1} 0 = S^1 S^m 0$

$$\boxed{\forall x\,(0 \leq x)}$$

P. Smith
§ 9.4
(01)

$$\frac{\dfrac{\dfrac{\dfrac{\overline{}\,\mathrm{Ax}}{x + 0 = x}}{\exists v\,(v + 0 = x)}}{0 \leq x}}{\forall x\,(0 \leq x)}$$

$$\boxed{x \leq \overline{k} \rightarrow x \leq \overline{k+1}}$$

P. Smith §9.4
(06)

$$[a + x = \overline{k}]^1$$

$$S(a + x) = S\overline{k}$$

$$a + Sx = S\overline{k}$$

Lemme auxiliaire

$$Sa + x = S\overline{k}$$

$$\exists y\,(y + x = S\overline{k})$$

$$[x \leq \overline{k}]^2$$

$$\exists v\,(v + x = \overline{k}) \qquad x \leq \overline{k+1} \quad 1$$

$$x \leq \overline{k+1} \quad 2$$

$$x \leq \overline{k} \rightarrow x \leq \overline{k+1}$$

$$\boxed{S x + \bar{n} = x + S\bar{n}}$$

Lemma ausiliario

$n = 0$

$$\dfrac{\dfrac{\dfrac{}{Sx + 0 = Sx}\,\text{Ax} \qquad \dfrac{}{x + 0 = x}\,\text{Ax}}{Sx + 0 = S(x + 0)}}{Sx + 0 = x + S0}\,\text{Ax}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Sx + \overline{n+1} = S(Sx + \bar{n})}\,\text{Ax} \qquad \dfrac{}{Sx + \bar{n} = x + S\bar{n}}\,\text{IH}}{Sx + \overline{n+1} = S(x + S\bar{n})}}{Sx + \overline{n+1} = x + SS\bar{n}}}{Sx + \overline{n+1} = x + S(\overline{n+1})}$$

$$\boxed{\bar{K} \leq \overline{K+1}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{S0 + \bar{K} = 0 + \overline{K+1}}\,\text{Lemma ausiliario}}{S0 + \bar{K} = \overline{K+1}}}{\exists v\,[v + \bar{K} = \overline{K+1}]}}{\bar{K} \leq \overline{K+1}}$$

Non vale in generale $0 + x = x$, ma vale per i numerali.

Vedi P. Smith § 8.2 e § 8.4

$$x \leq \overline{k} \longrightarrow (x = 0 \vee x = \overline{1} \vee \ldots \vee x = \overline{k})$$

Mostriamo che $x \leq 0 \rightarrow (x = 0)$

$$\frac{\dfrac{\dfrac{[x = Sb]^{1} \quad [a + x = 0]^{4}}{a + Sb = 0}}{S(a+b) = 0}\ \text{ax 1}}{\bot}$$

$$\frac{[\exists y\,(x = Sy)]^{3} \qquad \dfrac{\bot}{x = 0}}{x = 0}\ 1$$

$$\frac{\dfrac{\text{Ax 3}}{x = 0 \vee \exists y\,(x = Sy)} \qquad [x = 0]^{2} \qquad x = 0}{x = 0}\ 2,3$$

$$\frac{\dfrac{[x \leq 0]^{5}}{\exists v\,(v + x = 0)} \qquad x = 0}{x = 0}\ 4$$

$$\frac{x = 0}{x \leq 0 \longrightarrow x = 0}\ 5$$

Chiswell prop 3.1

P. Smith § 9.4 (03)

1. H $\quad x \leq \overline{k} \to (x = \overline{0} \vee x = \overline{1} \vee \dots \vee x = \overline{k})$

$$[x \leq \overline{k+1}]^4$$

$$\dfrac{[x = Sb]^1 \qquad \dfrac{[x \leq \overline{k+1}]^4}{\exists v\,(v + x = S\overline{k})}}{\dfrac{\exists v\,(v + Sb = S\overline{k})}{\dfrac{\exists v\,(S(v+b) = S\overline{k})}{\dfrac{\exists v\,(v + b = \overline{k})}{\dfrac{b \leq \overline{k}}{b = 0 \vee b = \overline{1} \vee \dots \vee b = \overline{k}}}}}}$$

$$\dfrac{[x = Sb]^1 \qquad Sb = S0 \vee Sb = S(\overline{1}) \vee \dots \vee Sb = S\overline{k}}{x = \overline{1} \vee x = \overline{2} \vee \dots \vee x = \overline{k+1}} \; 1$$

$$\dfrac{[\exists y(x = Sy)]^3 \qquad x = \overline{1} \vee x = \overline{2} \vee \dots \vee x = \overline{k+1}}{x = \overline{1} \vee \dots \vee x = \overline{k+1}}$$

$$\dfrac{[x = 0]^2}{x = 0 \vee \dots \vee x = \overline{k+1}} \qquad \dfrac{x = \overline{1} \vee \dots \vee x = \overline{k+1}}{x = 0 \vee x = \overline{1} \vee \dots \vee x = \overline{k+1}}$$

$$\dfrac{x = 0 \vee \exists y(x = Sy) \qquad x = 0 \vee \dots \vee x = \overline{k+1} \qquad x = 0 \vee x = \overline{1} \vee \dots \vee x = \overline{k+1}}{x = 0 \vee x = \overline{1} \vee \dots \vee x = \overline{k+1}} \; 2,3$$

$$\dfrac{x = 0 \vee x = \overline{1} \vee \dots \vee x = \overline{k+1}}{x \leq \overline{k+1} \to x = 0 \vee x = \overline{1} \vee \dots \vee x = \overline{k+1}} \; 4$$

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$$\bar{n} \leq x \to (\bar{n} = x \vee S\bar{n} = x)$$

P. Smith
§ 9.4
(07)

$$\dfrac{[a = Sb]^1 \quad [a + n = x]^4}{\dfrac{Sb + n = x}{\dfrac{b + Sn = x}{\dfrac{\exists v\,(v + Sn = x)}{Sn \leq x}}}}$$

$$\dfrac{[a + n = x]^4 \quad [a = 0]^3}{\dfrac{0 + n = x}{n = x}}$$

$$\dfrac{\exists y\,(a = Sy) \quad Sn \leq x}{Sn \leq x}\ 1$$

$$\dfrac{\dfrac{}{a = 0 \vee \exists y\,(a = Sy)}\,Ax \qquad \dfrac{n = x}{n = x \vee Sn \leq x} \qquad \dfrac{Sn \leq x}{n = x \vee Sn \leq x}}{n = x \vee Sn \leq x}\ 2, 3$$

$$\dfrac{\dfrac{[n \leq x]^5}{\exists v\,(v + n = x)} \qquad n = x \vee Sn \leq x}{\dfrac{n = x \vee Sn \leq x}{n \leq x \to n = x \vee Sn \leq x}\ 5}\ 4$$

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3 Teor

$[\bar{k} = a]^3$ $\quad$ $\bar{k} \leq \overline{k+1}$

$a = \bar{k}$ $\quad$ $a = \bar{k} \to a \leq \overline{k+1}$

(06) $\quad$ $[\bar{k} \leq a]^2$ $\quad$ (07) Teor $\quad$ $a \leq \overline{k+1}$ $\quad$ $[\overline{k+1} \leq a]^4$

1 Teor

$[a \leq \bar{k}]^1$ $\quad$ $a \leq \bar{k} \to a \leq \overline{k+1}$ $\quad$ $\bar{k} \leq a \to \bar{k} = a \vee \overline{k+1} \leq a$ $\quad$ $a \leq \overline{k+1} \vee \overline{k+1} \leq a$ $\quad$ $a \leq \overline{k+1} \vee \overline{k+1} \leq a$

$\bar{k} = a \vee \overline{k+1} \leq a$ $\quad$ $a \leq \overline{k+1} \vee \overline{k+1} \leq a$

$a \leq \overline{k+1}$ $\quad$ 3, 4

5 $\quad$ $[a \leq \bar{k} \vee \bar{k} \leq a]^5$ $\quad$ $a \leq \overline{k+1} \vee \overline{k+1} \leq a$ $\quad$ $a \leq \overline{k+1} \vee \overline{k+1} \leq a$

1, 2

$a \leq \overline{k+1} \vee \overline{k+1} \leq a$

Teor

$\forall x (0 \leq x)$

$0 \leq a$ $\quad$ 5

$a \leq \bar{k} \vee \bar{k} \leq a \longrightarrow a \leq \overline{k+1} \vee \overline{k+1} \leq a$

$a \leq 0 \vee 0 \leq a$

$\forall x (x \leq \bar{n} \vee \bar{m} < x)$

P. Smith § 8.4

$$0+\bar{n} = \bar{n}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{}\,\scriptstyle Ax}{0+0=0} \quad \dfrac{\overline{}\,\scriptstyle Ax}{0+S0=S(0+0)}}{0+S0=S0} \quad \dfrac{\overline{}\,\scriptstyle Ax}{0+SS0=S(0+S0)}}{0+SS0=SS0} \quad \dfrac{\overline{}\,\scriptstyle Ax}{0+SSS0=S(0+SS0)}}{0+SSS0=SSS0}$$

Non si dimostra che $0+x=x$!

Non possiamo usare l'induzione su x, perché non abbiamo l'assioma di induzione

Proviamo ad usare il teorema $x=0 \vee \exists y\,(x=Sy)$.

$$\dfrac{x=0 \qquad 0+x=\,?}{?}$$

non possiamo eseguire la sostituzione

Costruiamo un modello $\mathcal{M}$ per l'aritmetica di Robinson

in cui $\mathcal{M} \not\models \forall x(0+x=x)$

$\mathbb{D} = \mathbb{N} \cup \{a, b\}$ $\quad \hat{0} = 0 \quad \hat{S}(n) = n+1 \quad \hat{S}(a) = a \quad \hat{S}(b) = b$

$\hat{+}$: $a \hat{+} n = a \quad b \hat{+} n = b \quad x \hat{+} a = b \quad x \hat{+} b = a$

$\hat{\cdot}$: $a \hat{\cdot} Sn = b \quad b \hat{\cdot} Sn = a \quad x \hat{\cdot} a = b \quad x \hat{\cdot} b = a$

$a \hat{\cdot} 0 = 0 \quad b \hat{\cdot} 0 = 0$

$\mathcal{M} \not\models \forall x(0+x=x)$ infatti $0+a=b$

$\leq$ è fortemente rappresentata in $\mathbb{R}$.

$$[\bar{a} + SS0 = S0]^1$$

$$S(a + S0) = S0$$

$$\dfrac{SS(a+0) = S0 \qquad a+0=a}{SSa = S0}$$

$$\dfrac{SSa = S0}{Sa = 0}$$

$$\dfrac{Sa = 0}{0 = Sa}$$

$$\dfrac{0 = Sa}{\bot}$$

$$\dfrac{\exists v[v + SS0 = S0]^2 \qquad \bot}{\bot}\ 1$$

$$\dfrac{\bot}{\neg \exists v[v + SS0 = S0]}\ 2$$

$\mathfrak{N} \not\models m \leq n \Rightarrow R \vdash \neg(\bar{m} \leq \bar{n})$

Esempio per $m = 2$ e $n = 1$

$\mathfrak{N} \models m \leq n \Rightarrow \mathfrak{N} \models m + k = n$ per qualche k

$R \vdash \bar{m} + \bar{k} = \bar{n}$

$\vdash \exists v[v + \bar{k} = \bar{n}]$

$\vdash \bar{m} \leq \bar{n}$

Th 9.2 $\mathcal{R}$ è <u>Σ_1 - completa</u>

Th 9.3 Sia φ una <u>Π_1 - sentence</u>

$$\mathcal{N} \models \varphi \iff \mathcal{R} \cup \{\varphi\} \text{ è consistente}$$

$$\iff \mathcal{R} \nvdash \neg\varphi$$

φ: <u>Goldbach's conjecture</u>

$$\forall x \{ (\psi(x) \wedge 4 \leq x) \longrightarrow (\exists y \leq x) \exists z (\leq x) (\chi(y) \wedge \chi(z) \wedge y + z = x) \}$$

pari $\psi(x) =_{df} (\exists v \leq x)(2 \times v = x)$

primo $\chi(x) =_{df} x \neq 1 \wedge (\forall u \leq x)(\forall v \leq x)(u \times v = x \to (u = 1 \vee v = 1))$

Ghilardi
pag 5

In PA si dimostra l'assioma 3 dell'aritmetica di Robinson

$$\dfrac{\dfrac{\overset{\text{Ax}}{Sx = Sx} \qquad \overset{\text{Teor}}{Sx = Sx \longrightarrow \exists y\,(Sx = Sy)}}{\exists y\,(Sx = Sy)}}{Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)}$$

$$\dfrac{Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)}{\big(x \neq 0 \longrightarrow \exists y\,(x = Sy)\big) \longrightarrow \big[Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)\big]}$$

$$\dfrac{\big(x \neq 0 \longrightarrow \exists y\,(x = Sy)\big) \longrightarrow \big[Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)\big]}{\forall x\,\Big[\big(x \neq 0 \longrightarrow \exists y\,(x = Sy)\big) \longrightarrow \big(Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)\big)\Big]}$$

$$\dfrac{\dfrac{\overset{\text{Teor}}{0 \neq 0 \to \exists y\,(0 = Sy)} \qquad \forall x\,\Big[\big(x \neq 0 \longrightarrow \exists y\,(x = Sy)\big) \longrightarrow \big(Sx \neq 0 \longrightarrow \exists y\,(Sx = Sy)\big)\Big]}{\forall x\,\big(x \neq 0 \longrightarrow \exists y\,(x = Sy)\big)}}{\forall x\,\big(x = 0 \vee \exists y\,(x = Sy)\big)}$$